Student Number: | | | | | | | | |

Teacher Name: ____________________

Penrith Selective High School

**2023** **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

# Mathematics Extension 2

**General Instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- Reference sheets are provided with this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

**Total marks: 100**

**Section I – 10 marks** (pages 2–5)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 6–12)
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

| | Multiple Choice | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | Total |
|---|---|---|---|---|---|---|---|---|
| Complex | 4, 5 /2 | /6 | /3 | /4 | | | /5 | /20 |
| Proof | 1, 6 /2 | | | | /5 | /5 | /6 | /18 |
| Integration | 1, 9 /2 | /3 | /5 | /3 | /4 | /2 | /3 | /22 |
| Vectors | 2, 8 /2 | | /6 | /3 | /6 | | /2 | /19 |
| Mechanics | 3, 10 /2 | /7 | | /5 | | /7 | | /21 |
| Total | /10 | /16 | /14 | /15 | /15 | /14 | /16 | /100 |

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given the statement:

"If a prime number greater than 3 can be represented by $6n \pm 1$ then $n$ is a positive integer."

Which of the following statements is its contrapositive?

A. If $6n \pm 1$ is not a prime number greater than 3, then $n$ is not a positive integer.

B. If $n$ is not a positive integer, then $6n \pm 1$ is a prime number greater than 3.

C. If $n$ is not a positive integer, then $6n \pm 1$ is not a prime number greater than 3.

D. If $6n \pm 1$ is a prime number greater than 3, then $n$ is not a positive integer.

2 The points $P$, $Q$ and $R$ are collinear where $\overrightarrow{OP} = \tilde{\imath} - \tilde{\jmath}$, $\overrightarrow{OQ} = -3\tilde{\jmath} - \tilde{k}$ and $\overrightarrow{OR} = 2\tilde{\imath} + m\tilde{\jmath} + n\tilde{k}$ for some constants $m$ and $n$.

Which of the following are possible values for $m$ and $n$?

A. $m = -1$ and $n = -1$

B. $m = -1$ and $n = 1$

C. $m = 1$ and $n = 1$

D. $m = 1$ and $n = -1$

3 A particle is moving in simple harmonic motion with displacement $x$ metres. Its acceleration $\ddot{x}$ is given by $\ddot{x} = -4x + 3$. What are the centre and period of the motion?

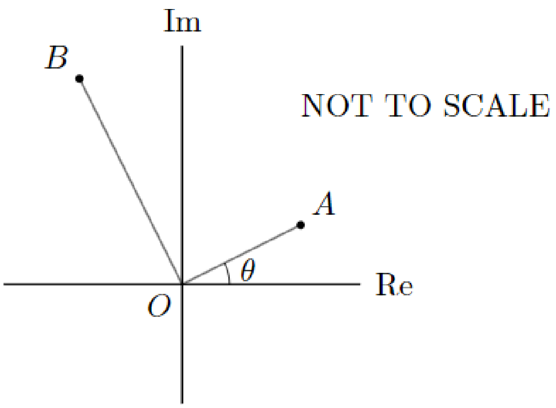
A. centre of motion = 3, period = $\frac{\pi}{2}$

B. centre of motion = $\frac{3}{4}$, period = $\pi$

C. centre of motion = − 3, period = $\pi$

D. centre of motion = $\frac{3}{4}$, period = $\frac{\pi}{2}$

4 The points $A$ and $B$ in the diagram represent the complex numbers $z_1$ and $z_2$ respectively, where $|z_1| = 1$ and Arg $(z_1) = \theta$ and $z_2 = \sqrt{3}\, iz_1$.




Which of the following represents $z_2 - z_1$?

A. $2e^{i(\frac{2\pi}{3} + \theta)}$

B. $3e^{i(\frac{2\pi}{3} + \theta)}$

C. $2e^{i(\frac{2\pi}{3} - \theta)}$

D. $3e^{i(\frac{2\pi}{3} - \theta)}$

5 Which of the following numbers is a $6^{\text{th}}$ root of $i$?

A. $\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, i$

B. $\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\, i$

C. $-\sqrt{2} + \sqrt{2}\, i$

D. $-\sqrt{2} - \sqrt{2}\, i$

6 Given the statement: "$\forall n \in \mathrm{Z}$, $n = 9m + 2 \Rightarrow n$ can be written as a sum of two square integers".

Which of the following statements is its negation?

A. $\forall n \in Z$, $n \neq 9m + 2$ and $n$ can be written as a sum of two square integers

B. $\exists n \in Z$, $n = 9m + 2$ and $n$ cannot be written as a sum of two square integers

C. $\forall n \in Z$, $n = 9m + 2$ and $n$ cannot be written as a sum of two square integers

D. $\exists n \in Z$, $n \neq 9m + 2$ and $n$ can be written as a sum of two square integers

7 By using the substitution $t = \tan\frac{x}{2}$, $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$ can be expressed as:

A. $\displaystyle\int_0^1 \frac{1}{1+2t}\,dt$

B. $\displaystyle\int_0^1 \frac{2}{1+2t}\,dt$

C. $\displaystyle\int_0^1 \frac{1}{(1+t)^2}\,dt$

D. $\displaystyle\int_0^1 \frac{2}{(1+t)^2}\,dt$

8 The scalar product of $5\tilde{\imath} + \tilde{\jmath} - 3\tilde{k}$ and $3\tilde{\imath} - 4\tilde{\jmath} + 7\tilde{k}$ is:

A. $10$

B. $-10$

C. $15$

D. $-15$

9 Which expression is equivalent to $\int \frac{dx}{\sqrt{x^2+1}}$ ?

A. $\ln\left|\sqrt{1+x^2}+x\right|+c$

B. $\ln\left|\sqrt{1+x^2}\right|+c$

C. $\ln\left|\sqrt{1+x^2}-x\right|+c$

D. $\ln\left|x\sqrt{1+x^2}\right|+c$

10 When Mr Kim applies his brakes in his car, the change in velocity is given by $\frac{dv}{dt}=kv$, where $k$ is a constant. Initially his car was travelling at 6 $ms^{-1}$. Ten seconds later, he was travelling at 2 $ms^{-1}$.

How fast was Mr Kim travelling five seconds after the brakes were applied?
(answer to three significant figures)

A. 3.29 $ms^{-1}$

B. 4.82 $ms^{-1}$

C. 4.72 $ms^{-1}$

D. 3.46 $ms^{-1}$

**Section II**
**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section.**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (16 marks) Use a separate Writing Booklet

(a) If $\omega$ is a complex cube root of unity, simplify each of the following:

(i) $\omega^5$ **1**

(ii) $1 + w^4 + \omega^8$ **2**

(b) Find the modulus and argument of $z = \frac{5-i}{2-3i}$. **3**

(c) Evaluate exactly as a fraction with a rational denominator: **3**

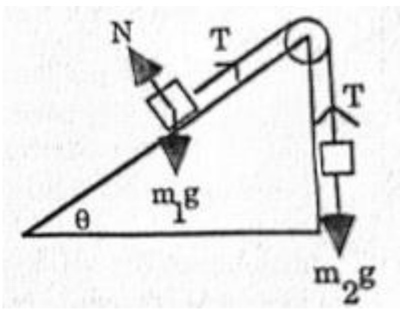
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cos^3 x}{sin^2 x}\, dx\ .$$

(d) The velocity of a particle in $m/s$ is moving in simple harmonic motion along the $x$-axis is given by $v^2 = -x^2 - 4x + 12$.

(i) State the centre and period of motion. **2**

(ii) What is the maximum acceleration of the particle? **2**

(e) Two masses, $m_1$ and $m_2$ are connected by a light inelastic string passing over a smooth pulley. The first mass, $m_1$ rests on s smooth plane which makes an angle of $\theta$ with the horizontal. The second mass hangs in the air. The tension in the string is T and N is the normal reaction of the plane on the mass $m_1$. If the masses are stationary, show that: $\frac{T^2}{{m_1}^2} + \frac{N^2}{{m_1}^2} = g^2$ **3**



**Question 12** (14 marks) Use a separate Writing Booklet

(a) Simplify $\dfrac{1+i+i^2+i^3+\ldots+i^{1000}}{1-i+i^2-i^3+\ldots+i^{1000}}$. **3**

(b) Find the values of $A$, $B$ and $C$ such that $\dfrac{1}{x(1+x^2)} \equiv \dfrac{A}{x}+\dfrac{Bx+C}{1+x^2}$ and hence evaluate $\int_{0.5}^{2} \dfrac{dx}{x(1+x^2)}$. **5**

Leave your answer in the form $\ln|b|$, where $b \in Z$.

(c) (i) Find the distance between the centres of the two spheres with equations: **2**

$\left|\tilde{r}-\begin{pmatrix}3\\-4\\3\end{pmatrix}\right| = 5$ and $\left|\tilde{r}-\begin{pmatrix}-7\\7\\1\end{pmatrix}\right| = 10$. What is the significance of your answer?

(ii) Find the coordinates of the point/s of contact between the two spheres. **2**

(iii) Find the vector equation of the line passing through the two centres. **2**

**Question 13** (15 marks) Use a separate Writing Booklet

(a) A particle is initially at rest on the number line at a position of $x = 1$. The particle moves continuously along the number line according to the acceleration equation:

$$\ddot{x} = \frac{4}{(x-2)^2} + \frac{8}{x^3}.$$

(i) At time $t$, the velocity of the particle is $v$. Show that **2**

$$v^2 = \frac{-8}{x-2} - \frac{8}{x^2}$$

(ii) Hence or otherwise, determine the possible range of the particle's displacement as it moves along the number line. **3**

(b) (i) Write the complex number $1 - \sqrt{3}\,i$ in exponential form. **2**

(ii) Hence find the exact value of $(1 - \sqrt{3}\,i)^8$ giving your answer in the form $x + yi$. **2**

(c) A triangle is formed in 3-D space with vertices $A(1, -2, 3), B(2, 0, 3)$ and $C(4, 2, 1)$. **3**

Find the size of $< ABC$, giving your answer to the nearest minute.

(d) Find $\int_0^{\frac{\pi}{4}} \cos^{-1} x \, dx$ correct to two decimal places. **3**

**Question 14** (15 marks) Use a separate Writing Booklet

(a) If $I_n = \int tan^n\ x\ dx$

(i) Show that $I_n + I_{n-2} = \frac{tan^{n-1} x}{n-1} + c.$ **3**

(ii) Hence, or otherwise, evaluate exactly $\int_0^{\frac{\pi}{4}} (tan^7 x + tan^5 x)\ dx.$ **1**

(b) It is given that $a + b + c = 1$ and $a + b + c = 3\sqrt[3]{abc}$ where $a$, $b$ and $c$ are positive real integers.

(i) Prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$ **3**

(ii) Hence or otherwise find the smallest possible value of $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right).$ **2**

(c) Given $A(1,1), B(2,8)$ and $C(-1,5)$ are the vertices of a triangle:

(i) Find the vector equation of the line passing through $A$ perpendicular to $BC$. **2**

(ii) The orthocentre of a triangle is the point of intersection of the three altitudes of the triangle. Find the coordinates of the orthocentre of the triangle $ABC$. **4**

**Question 15** (14 marks) Use a separate Writing Booklet

(a) (i) Show that, if $0 < x < \frac{\pi}{2}$, then $\frac{\sin(5a+3)x}{\sin 3x} - \frac{\sin(5a-3)x}{\sin 3x} = 2\cos(5ax)$. **2**

(ii) Deduce that, if $a$ is any integer then, $\int_0^{\frac{\pi}{5}} \frac{\sin(5a+3)x}{\sin 3x}\, dx = \int_0^{\frac{\pi}{5}} \frac{\sin(5a-3)x}{\sin 3x}\, dx$. **2**

(b) In a certain sequence $T_n$, $T_1 = 3, T_2 = 5$ and $T_{n+2} = 4T_{n+1} - 3T_n$. **3**

Prove by mathematical induction that $T_n = 3^{n-1} + 2$.

(c) The clearance for shipping under the Sydney Harbour Bridge is 45 metres. The Penrith Sun cruise ship is a luxury cruise ship which requires 43.8 metres height above the water level to safely cruise under the bridge. The process of sailing under the bridge will take 15 minutes. The first low tide on Saturday the 26th of August was at 2 am and the first high tide was at 8 am. The depth of water at low tide was 0.4 m and at high tide it was 1.5 m. Assume that the tidal motion is simple harmonic motion.

(i) Neatly draw the displacement graph showing two complete wavelengths for the Penrith Sun cruise ship starting from 2am on Saturday the 26th of August. **2**

(ii) Hence show that the displacement equation can be written in the form $x = -b\cos(nt) + c$. **2**

(iii) Determine between what times on Saturday can the Penrith Sun cruise ship safely make the passage under the bridge, given that no ships are allowed under the Sydney Harbour Bridge between 8 pm and 2 am? **3**

**Question 16** (16 marks) Use a separate Writing Booklet

(a) What is the projection vector of the vector $2\tilde{\imath} + 3\tilde{\jmath} - 6\tilde{k}$ on the line joining the points $(3,4,2)$ and $(5,6,3)$? **2**

(b) (i) If $z = \cos\theta + i\sin\theta$, show that $\sin n\theta = \frac{1}{2i}(z^n - \frac{1}{z^n})$. **2**

(ii) Express $sin^5\theta$ in terms of multiple angles. **2**

(iii) Hence find $\int sin^5\theta \, d\theta$. **1**

(c) Evaluate exactly $\int_{\sqrt{3}}^{3} \frac{6x+18}{x^2+9} \, dx$. **3**

(d) (i) Prove that $\cos x = 1 - 2sin^2\frac{x}{2}$ **1**

(ii) Prove by mathematical induction that $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$ for all $n \in Z^+$. **5**

# End of examination!

M/C 1. C 2. C 3. B 4. A 5. A
6. B 7. D 8. B 9. A 10. D

**Question 11 (16 marks)**

**(a)** If $\omega$ is a complex cube root of unity, simplify each of the following:

(i) $\omega^5$ **1**

$\omega^3 = 1$ (given)

$\therefore \omega^5 = \omega^3 \times \omega^2$

$= 1 \times \omega^2$

$\Rightarrow \omega^5 = \omega^2$

- Poor attempt
- students were finding roots of unity and finding the value of $\omega^2$.

(ii) $1 + w^4 + \omega^8$ **2**

$\omega^3 = 1$ (given)

$\therefore \omega^3 - 1 = 0$

$(\omega - 1)(1 + \omega + \omega^2) = 0$

$\Rightarrow 1 + \omega + \omega^2 = 0$ as $\omega \neq 1$ complex root of unity.

$1 + \omega^3 \times \omega + \omega^6 \times \omega^2$

$= 1 + \omega + \omega^2$

$= 0$

- poor attempt
- students were discarding $(\omega - 1)$ without reasoning

**(b)** Find the modulus and argument of $z = \frac{5-i}{2-3i}$. **3**

$z = \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$ (realising the denominator)

$= \frac{10 + 15i - 2i - 3i^2}{4 - 9i^2}$

$= \frac{13 + 13i}{13}$ $(i^2 = -1)$

$= 1 + i$

$|z| = \sqrt{1+1}$

$= \sqrt{2}$

$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$

$\arg z = \pi/4$

- students did okay in this question, apart from some algebra mistakes

**(c)** Evaluate exactly as a fraction with a rational denominator $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sin^2 x}\,dx$ **3**

$$I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x \, dx}{\sin^2 x}$$

$$= \int_{\pi/6}^{\pi/3} \frac{(1-\sin^2 x)\cos x \, dx}{\sin^2 x}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos x \, dx}{\sin^2 x} - \int_{\pi/6}^{\pi/3} \cos x \, dx$$

$$= \left[-\frac{1}{\sin x}\right]_{\pi/6}^{\pi/3} - \left[\sin x\right]_{\pi/6}^{\pi/3}$$

$$= -\left[\frac{2}{\sqrt{3}} - 2\right] - \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right]$$

$$= \frac{15 - 7\sqrt{3}}{6}$$

- many students could integrate it correctly, but many lost marks while evaluating the integrand.

**(d)** The velocity of a particle in $m/s$ is moving in simple harmonic motion along the $x$-axis is given by $v^2 = -x^2 - 4x + 12$.

(i) State the centre and period of motion. **2**

(ii) What is the maximum acceleration of the particle? **2**

(i)

$$v^2 = -(x^2 + 4x + 4) + 4 + 12$$

$$= -((x+2)^2 - 16)$$

or

$$v^2 = 16 - (x+2)^2$$

in SHM,

$$v^2 = n^2(a^2 - (x - x_0)^2)$$

$\therefore n = 1$, $T = \frac{2\pi}{n}$

$\therefore$ Time Period $= 2\pi$

Centre $x_0 = -2$

(ii) At extremes, $v = 0$

$\Rightarrow x = -6$ and $2$

max acceleration at the extremes

$\ddot{x} = -(-6) - 2 = 4\,m/s^2$, $\ddot{x} = -(2) - 2 = -4\,m/s^2$

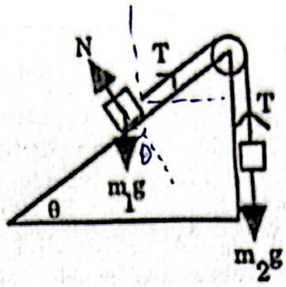
$\therefore$ max acc $= 4\,m/s^2$

- some students wrote $-4\,m/s^2$ as max.
- algebra errors
- lost 1 mark for using $v=0$ without reasoning.

**(c)** Two masses, $m_1$ and $m_2$ are connected by a light inelastic string passing over a smooth pulley. **3**

The first mass, $m_1$ rests on a smooth plane which makes an angle of $\theta$ with the horizontal. The second mass hangs in the air. The tension in the string is T and N is the normal reaction of the plane on the mass $m_1$. If the masses are stationary, show that:

$$\frac{T^2}{m_1^2} + \frac{N^2}{m_1^2} = g^2$$




masses stationary

$\Rightarrow F_{net} = (F_{net})_{\text{parallel forces}}$

$= (F_{net})_{\text{perpendicul. forces}}$

$= 0.$

$(F_{net})_{parallel}$ : $T - m_1 g \sin\theta = 0$

$(F_{net})_{\perp}$ : $N - m_1 g \cos\theta = 0$

$\Sigma F = 0$

$T = m_2 g$

$\therefore m_2 g = m_1 \sin\theta g$ (same tension)

$m_2 = m_1 \sin\theta$

$$LHS = \frac{T^2}{m_1^2} + \frac{N^2}{m_1^2}$$

$$= \frac{\cancel{m_1^2} g^2 \sin^2\theta}{\cancel{m_1^2}} + \frac{\cancel{m_1^2} g^2 \cos^2\theta}{\cancel{m_1^2}}$$

$$= g^2 \sin^2\theta + g^2 \cos^2\theta$$

$$= g^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{= 1}$$

$$= g^2$$

$= RHS.$

Poor attempt :

**End of Question 11**

**Questions 12**

Done well **(a)** Simplify $\dfrac{1+i+i^2+i^3+\ldots+i^{1000}}{1-i+i^2-i^3+\ldots+i^{1000}}$. **3**

$$= \frac{\dfrac{1(i^{1001}-1)}{i-1}}{\dfrac{1((-i)^{1001}-1)}{-i-1}} \quad ①$$

C/M $\dfrac{1(i^{1000}-1)}{i-1}$ There are 1001 terms.

$$= \frac{i^{1001}-1}{i-1} \times \frac{-i-1}{i^{1001}-1}$$

$$= \frac{i-1}{i-1} \times \frac{-i-1}{-i-1} \quad \left(\text{Since } i^{1001} = i \times i^{1000} = i \times (i^2)^{500} = i \times (-1)^{500} = i\right) \quad ①$$

$$= 1 \quad ①$$

**Alternative 1:**

Since $i^2=-1,\ i^3=-i,\ i^4=1,\ i^5=i,\ i^6=-1, \ldots$

$$\frac{(1+i+i^2+i^3)+(i^4+i^5+i^6+i^7)+\cdots+i^{1000}}{(1-i+i^2-i^3)+(i^4-i^5+i^6-i^7)+\cdots+i^{1000}}$$

$$= \frac{(1+i-1-i)+(1+i-1-i)+\cdots+1}{(1-i-1+i)+(1-i-1+i)+\cdots+1}$$

$$= \frac{1}{1}$$

$$= 1$$

**Alternative 2:**

$$\frac{1+i+i^2+i^3+\cdots+i^{1000}}{1+i^2 i+i^2+i^2 i^3+i^4+\cdots+i^{1000}}$$

$$= \frac{1+i+i^2+i^3+\cdots+i^{1000}}{1+i^2+i^3+i^4+i^5+\cdots+i^{1000}+i^{1001}}$$

$$= \frac{1+(i+i^2+i^3+i^4)+i^4(i+i^2+i^3+i^4)+\cdots+i^{996}(i+i^2+i^3+i^4)}{1+i(i+i^2+i^3+i^4)+i^3(i+i^2+i^3+i^4)+\cdots+i^{997}(i+i^2+i^3+i^4)}$$

$$= \frac{1}{1} \quad \left(\text{Since } i+i^2+i^3+i^4 = i-1-i+1 = 0\right)$$

$$= 1$$

**Questions 12**

Done well

**(b)** Find the values of $A$, $B$ and $C$ such that $\frac{1}{x(1+x^2)} \equiv \frac{A}{x} + \frac{Bx+C}{1+x^2}$ and hence evaluate: **5**

$$\int_{0.5}^{2} \frac{dx}{x(1+x^2)}$$

Leave your answer in the form $\ln|b|$, where $b \in Z$.

$1 = A(1+x^2) + (Bx+C)x$

Sub. $x=0$, $\therefore A = 1$

Sub. $x=1$, $1 = 2A + B + C$

$B + C = -1$ [1]

Sub. $x=-1$, $1 = 2A + B - C$

$B - C = -1$ [2]

[1] + [2], $2B = -2$

$\therefore B = -1$

[1] − [2], $2C = 0$

$\therefore C = 0$

2

$\therefore \frac{1}{x(1+x^2)} \equiv \frac{1}{x} - \frac{x}{1-x^2}$

$$\int_{0.5}^{2} \frac{dx}{x(1+x^2)} = \int_{0.5}^{2} \left(\frac{1}{x} - \frac{x}{1-x^2}\right) dx$$

$$= \left[\ln|x| - \tfrac{1}{2}\ln(1+x^2)\right]_{0.5}^{2} \quad 1$$

$$= \left(\ln 2 - \tfrac{1}{2}\ln 5\right) - \left(\ln\tfrac{1}{2} - \tfrac{1}{2}\ln\tfrac{5}{4}\right)$$

$$= \ln 2 - \ln\sqrt{5} - \left(\ln 2^{-1} - \ln\tfrac{\sqrt{5}}{2}\right)$$

$$= \cancel{\ln 2} - \cancel{\ln\sqrt{5}} + \ln 2 + \cancel{\ln\sqrt{5}} - \cancel{\ln 2}$$

1

$$= \ln 2 \quad 1$$

**Questions 12**

**(c)** (i) Find the distance between the centres of the two spheres with equations: **2**

$$\left|\underset{\sim}{r} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}\right| = 5 \text{ and } \left|\underset{\sim}{r} - \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix}\right| = 10.$$ What is the significance of your answer?

$C_1(3,-4,3)$, $C_2(-7,7,1)$

$$\overrightarrow{C_1C_2} = \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix}$$

$$\left|\overrightarrow{C_1C_2}\right| = \sqrt{(-10)^2 + 11^2 + (-2)^2} = \sqrt{225} = 15 \quad (1)$$

The distance between the centres equals the sum of the two radii. Thus, there is only 1 point of contact between the spheres. (1)

(ii) Find the coordinates of the point/s of contact between the two spheres. **2**

$\quad 5 \qquad\qquad 10$

$C_1(3,-4,3) \qquad P(x,y,z) \qquad C_2(-7,7,1)$

$$\overrightarrow{C_1P} = \frac{1}{3}\overrightarrow{C_1C_2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} \\ \frac{11}{3} \\ -\frac{2}{3} \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{7}{3} \end{pmatrix}$$

$\therefore \left(-\frac{1}{3}, -\frac{1}{3}, \frac{7}{3}\right)$ (1)

**Alternative:**

$$\overrightarrow{C_2P} = -2\overrightarrow{C_1P}$$

$$\begin{pmatrix} x+7 \\ y-7 \\ z-1 \end{pmatrix} = -2\begin{pmatrix} x-3 \\ y+4 \\ z-3 \end{pmatrix}$$

$x+7 = -2(x-3)$, $y-7 = -2(y+4)$, $z-1 = -2(z-3)$

$3x = -1$, $3y = -1$, $3z = 7$

$x = -\frac{1}{3}$, $y = -\frac{1}{3}$, $z = \frac{7}{3}$

$\therefore \left(-\frac{1}{3}, -\frac{1}{3}, \frac{7}{3}\right)$

C/M

* Attempted to solve simultaneously $(x-3)^2 + (y+4)^2 + (z-3)^2 = 25$, $(x+7)^2 + (y-7)^2 + (z-3)^2 = 100$ but failed to solve.
* $\overrightarrow{C_2P} \neq 2\overrightarrow{C_1P}$

(iii) Find the vector equation of the line passing through the two centres. **2**

$$\overrightarrow{C_1C_2} = \begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{r} = \overrightarrow{OC_1} + \lambda\left(\overrightarrow{C_1C_2}\right)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix} \quad (2)$$

**Questions 13**

**(a)** A particle is initially at rest on the number line at a position of $x = 1$.
The particle moves continuously along the number line according to the acceleration equation:

$$\ddot{x} = \frac{4}{(x-2)^2} + \frac{8}{x^3}.$$

Done well (i) At time $t$, the velocity of the particle is $v$. Show that $v^2 = \frac{-8}{x-2} - \frac{8}{x^2}$ **2**

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{(x-2)^2} + \frac{8}{x^3}$$

$$\frac{1}{2}v^2 = \int \left(4(x-2)^{-2} + 8x^{-3}\right) dx$$

$$\frac{1}{2}v^2 = -4(x-2)^{-1} - 4x^{-2} + c \quad (1)$$

Sub. $x=1$, $v=0$

$$0 = -4(-1)^{-1} - 4(1)^{-2} + c$$

$$\therefore c = 0$$

$$\frac{1}{2}v^2 = -\frac{4}{x-2} - \frac{4}{x^2} \quad (1)$$

$$\therefore v^2 = -\frac{8}{x-2} - \frac{8}{x^2}$$

Poorly done (ii) Hence or otherwise, determine the possible range of the particle's displacement as it moves along the number line. **3**

$v = \sqrt{\frac{-8}{x-2} - \frac{8}{x^2}}$ ($v \geq 0$ since $\ddot{x} > 0$ and the particle was initially at $x=1$ with $v=0$)

$\frac{-8}{x-2} - \frac{8}{x^2} \geq 0$, $x \neq 2$, $x \neq 0$

Multiply both sides by $x^2(x-2)^2$,

$-8x^2(x-2) - 8(x-2)^2 \geq 0$ (1)

$x^2(x-2) + (x-2)^2 \leq 0$

$(x-2)(x^2+x-2) \leq 0$

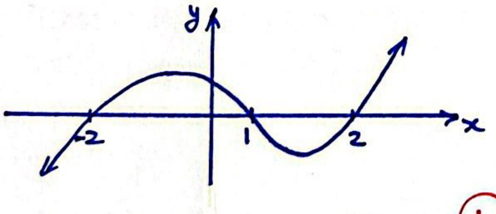
$(x-2)(x+2)(x-1) \leq 0$

C/M

* $\frac{-8x^2 - 8(x-2)}{x^2(x-2)} \geq 0$

$-8x^2 - 8(x-2) \not\geq 0$

Is $x-2$ positive for all $x \geq 1$?




$x \leq -2$ or $1 \leq x < 2$ (1)

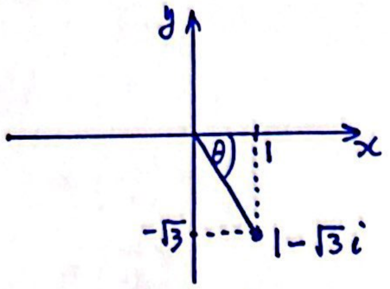
But the particle was initially at $x=1$ and moving to the right since $\ddot{x} = \frac{4}{(1-2)^2} + \frac{8}{(1)^3} = 12 > 0$.

$\therefore 1 \leq x < 2$ (1)

**Questions 13**

Done Well

**(b)** (i) Write the complex number $1 - \sqrt{3}i$ in exponential form. **2**




$r^2 = 1^2 + (\sqrt{3})^2$, $\tan\theta = \sqrt{3}$

$r = \sqrt{4}$ $\theta = \frac{\pi}{3}$ ①

$r = 2$

$\therefore 1 - \sqrt{3}i = 2e^{-\frac{\pi}{3}i}$ ①

(ii) Hence find the exact value of $(1 - \sqrt{3}i)^8$ giving your answer in the form $x + yi$. **2**

$$(1-\sqrt{3}i)^8 = (2e^{-\frac{\pi}{3}i})^8$$
$$= 2^8 e^{-\frac{8\pi}{3}i} \quad \text{(de Moivre's theorem)} \ ①$$
$$= 2^8\left(\cos\left(-\frac{8\pi}{3}\right) + i\sin\left(-\frac{8\pi}{3}\right)\right)$$
$$= 2^8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= 2^7(-1 - \sqrt{3}i)$$
$$= -128 - 128\sqrt{3}i \ ①$$

**Questions 13**

**(c)** A triangle is formed in 3-D space with vertices $A(1,-2,3), B(2,0,3)$ and $C(4,2,1)$. **3**

Find the size of $\angle ABC$, giving your answer to the nearest minute.

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ , $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \begin{pmatrix}1\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\0\\3\end{pmatrix}$ $\quad = \begin{pmatrix}4\\2\\1\end{pmatrix} - \begin{pmatrix}2\\0\\3\end{pmatrix}$

$= \begin{pmatrix}-1\\-2\\0\end{pmatrix}$ $\quad = \begin{pmatrix}2\\2\\-2\end{pmatrix}$ 1

$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$

$-2-4+0 = \sqrt{1+4+0}\sqrt{4+4+4}\cos\theta$ 1

$-6 = \sqrt{5}\sqrt{12}\cos\theta$

$\cos\theta = -\frac{6}{2\sqrt{15}}$

$\therefore \theta = 140°46'$ 1

C/M

* $\overrightarrow{AB} \cdot \overrightarrow{BC} = |\overrightarrow{AB}||\overrightarrow{BC}|\cos\theta$

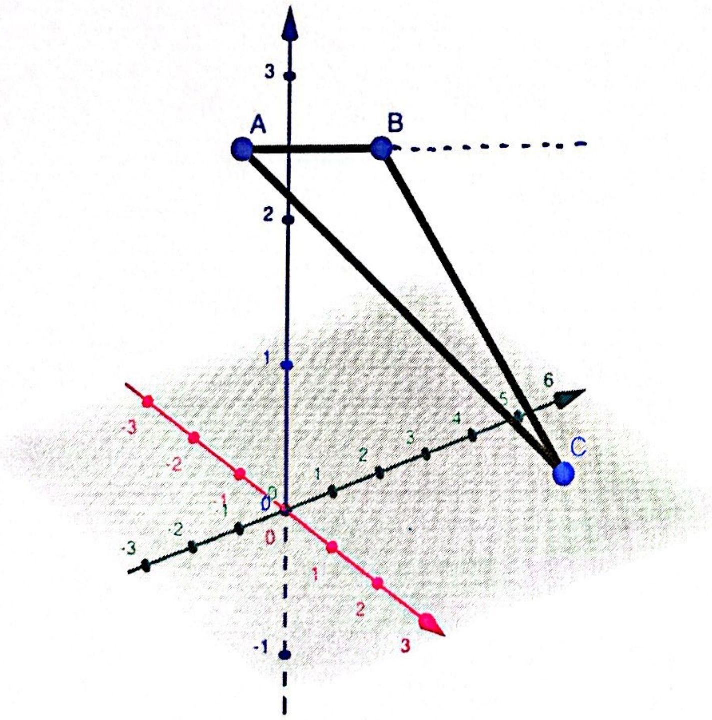
$2+4+0 = \sqrt{5}\sqrt{12}\cos\theta$

$\cos\theta = \frac{6}{2\sqrt{15}}$

$\theta = 39°14'$

* $\cos\theta = -\frac{6}{2\sqrt{15}}$

$\theta = 2°27'$ ← Check the mode of your calculator.



**Questions 13**

**(d)** Find $\displaystyle\int_0^{\frac{\pi}{4}} \cos^{-1} x\, dx$ correct to two decimal places. **3**

$u = \cos^{-1} x$ $\quad v' = 1$

$u' = -\dfrac{1}{\sqrt{1-x^2}}$ $\quad v = x$

$$\int_0^{\frac{\pi}{4}} \cos^{-1} x\, dx = \left[x\cos^{-1}x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{x}{\sqrt{1-x^2}}\,dx \quad \text{(by parts)} \quad ①$$

$$= \left(\frac{\pi}{4}\cos^{-1}\frac{\pi}{4} - 0\right) - \frac{1}{2}\int_0^{\frac{\pi}{4}} (-2)x(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= \frac{\pi}{4}\cos^{-1}\frac{\pi}{4} - \frac{1}{2}\left[2(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{\pi}{4}} \quad ①$$

$$= \frac{\pi}{4}\cos^{-1}\frac{\pi}{4} - \left(\sqrt{1-\frac{\pi^2}{16}} - 1\right)$$

$$= 0.91 \quad ①$$

C/M

* 30.42 ?

Check the calculator mode.

* $\cos^{-1}\frac{\pi}{4} \neq \frac{1}{\sqrt{2}}$

**Question 14 (15 marks)**

**(a)** If $I_n = \int tan^n x\, dx$

(i) Show that $I_n + I_{n-2} = \frac{tan^{n-1}x}{n-1} + c.$ **2**

Method I

$$I_n + I_{n-2} = \int tan^n x\, dx + \int tan^{n-2} x\, dx$$

$$= \int tan^{n-2} x\,(tan^2 x + 1)\, dx$$

$$= \int tan^{n-2} x\, sec^2 x\, dx$$

$$= \frac{tan^{n-1} x}{n-1} + c$$

$$\boxed{\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c}$$

Method II

$$I_n = \int tan^{n-2} x \cdot tan^2 x\, dx$$

$$= \int tan^{n-2} x \cdot (sec^2 x - 1)\, dx$$

$$= \int tan^{n-2} x\, sec^2 x\, dx - I_{n-2}$$

$$I_n + I_{n-2} = \int tan^{n-2} x\, sec^2 x\, dx$$

$$= \frac{tan^{n-1} x}{n-1} + c$$

Comments: - Majority of students used Method II

- this question was done well, demonstrating their ability to apply indices.

(ii) Hence, or otherwise, evaluate exactly **2**

$$\int_0^{\frac{\pi}{4}} (tan^7 x + tan^5 x)\, dx$$

using part a(i), for $n = 7$.

$$\int_0^{\pi/4} (tan^7 x + tan^5 x)\, dx = \left[\frac{tan^6 x}{6}\right]_0^{\pi/4} \quad \text{— 1 mark}$$

$$= \frac{1}{6}\left((\tan \pi/4)^6 - 0\right)$$

$$= \frac{1}{6} \quad \text{— 1 mark}$$

Comments: Majority of students got full marks, still, the students were doing it in a long way.

- Very few were struggling to evaluate the integrand

Method II $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)=\left(\frac{1-a}{a}\right)\left(\frac{1-b}{b}\right)\left(\frac{1-c}{c}\right)$

$=\left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right)\left(\frac{a+b}{c}\right)$ $a+b+c=1$

$\geq \frac{2\sqrt{bc}\times 2\sqrt{ac}\times 2\sqrt{ab}}{abc} = 8.$

**(b)** It is given that $a+b+c=1$ and $a+b+c=3\sqrt[3]{abc}$ where $a$, $b$ and $c$ are positive real integers.

**(i)** Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9$. 3

- students were manipulating the given data
- majority of students were writing $3^3=9$
- Poor attempt.

$1 \geq 3\sqrt[3]{abc}$

$\frac{1}{3} \geq \sqrt[3]{abc}$

$\frac{1}{\sqrt[3]{abc}} \geq 3$ — ①

Let $x=\frac{1}{a}$, $y=\frac{1}{b}$, $z=\frac{1}{c}$

Since $x+y+z = 3\sqrt[3]{xyz}$

$\geq 3\sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}$

$\geq 3\times\frac{1}{\sqrt[3]{abc}}$

$\geq 3\times 3$

$\geq 9$

**(ii)** Hence or otherwise find the smallest possible value of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$. 2

$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)=\left(\frac{1}{ab}-\frac{1}{a}-\frac{1}{b}+1\right)\left(\frac{1}{c}-1\right)$

$=\frac{1}{abc}-\frac{1}{ab}-\frac{1}{ac}+\frac{1}{a}-\frac{1}{bc}+\frac{1}{b}+\frac{1}{c}-1$

$=\frac{1}{abc}-\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1$

$=\frac{1}{abc}-\left(\frac{a+b+c}{abc}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1$

$=\cancel{\frac{1}{abc}}-\cancel{\frac{1}{abc}}+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1$ $a+b+c=1$

$\geq 9-1$

$=8$

smallest possible value $=8$.

- students were expanding it correctly, but did not know what to do afterwards,
- some were successful, but many were lost and making minor mistakes

**(c)** Given $A(1,1)$, $B(2,8)$ and $C(-1,5)$ are the vertices of a triangle:

**(i)** Find the vector equation of the line passing through $A$ perpendicular to $BC$. **2**

$m_{\overrightarrow{BC}} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \begin{pmatrix} -1-2 \\ 5-8 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

$\perp m_{\overrightarrow{BC}} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

or $\perp m_{\overrightarrow{BC}} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$\therefore$ the vector eqn of the line passing through $\underset{\sim}{A}$ and $\perp \overrightarrow{BC}$ is

$\underset{\sim}{r} = \underset{\sim}{A} + \lambda(\perp \overrightarrow{BC}), \lambda \in \mathbb{R}$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

or

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

comments:
- Many students were using $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ rather than the $\perp$ direction vector
- students are suggested to check $m_1 \times m_2 = -1$ the basic property for two lines to be perpendicular.

**(ii)** The orthocentre of a triangle is the point of intersection of the three altitudes of the triangle. Find the coordinates of the orthocentre of the triangle $ABC$. **4**

$\overrightarrow{AB} = \begin{pmatrix} 2-1 \\ 8-1 \end{pmatrix}$

$m_{\overrightarrow{AB}} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \Rightarrow \perp m_{AB} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -1-2 \\ 5-8 \end{pmatrix}$

$m_{\overrightarrow{BC}} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \Rightarrow \perp m_{\overrightarrow{BC}} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$\overrightarrow{CA} = \begin{pmatrix} 1--1 \\ 1-5 \end{pmatrix}$

$m_{\overrightarrow{CA}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \Rightarrow \perp m_{\overrightarrow{CA}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

vector eqn of three altitudes

$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ $\quad \lambda, \mu, \Omega \in \mathbb{R}$

$\underset{\sim}{r_2} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -7 \\ 1 \end{pmatrix}$

$\underset{\sim}{r_3} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} + \Omega \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

For Point of intersection:

$\left.\begin{aligned} 1+3\lambda &= -1-7\mu \\ 1-3\lambda &= 5+\mu \end{aligned}\right\}$

adding $-6\mu + 4 = 2$

$\Rightarrow \boxed{\mu = \frac{1}{3}}$ $\therefore \boxed{\lambda = -\frac{13}{9}}$

sub in third line

$2 + 4\Omega = -1 - 7\mu$

$\Rightarrow \boxed{\Omega = -\frac{4}{3}}$

$\therefore$ Point of intersection $= \begin{pmatrix} -1 \\ 5 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -7 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -\frac{10}{3} \\ \frac{16}{3} \end{pmatrix}$

comments: There were a number of mistakes, including using direction vector instead of $\perp$ direction vector, algebraic calculations.

**End of Question 14**

- There was only one student, who used the graphical method to find the orthocentre.

**Questions 15**

Done well

(a) (i) Show that, if $0 < x < \frac{\pi}{2}$, then $\frac{\sin(5a+3)x}{\sin 3x} - \frac{\sin(5a-3)x}{\sin 3x} = 2\cos(5ax)$. **2**

$$LHS = \frac{\sin(5a+3)x - \sin(5a-3)x}{\sin 3x}$$

$$= \frac{\sin(5ax+3x) - \sin(5ax-3x)}{\sin 3x} \quad ①$$

$$= \frac{2\cos 5ax \cancel{\sin 3x}}{\cancel{\sin 3x}} \quad (\text{Product to sum}) \quad ①$$

$$= 2\cos 5ax$$

$$= RHS$$

(ii) Deduce that, if $a$ is any integer then, **2**

$$\int_0^{\frac{\pi}{5}} \frac{\sin(5a+3)x}{\sin 3x}\,dx = \int_0^{\frac{\pi}{5}} \frac{\sin(5a-3)x}{\sin 3x}\,dx$$

Consider $\int_0^{\frac{\pi}{5}} \frac{\sin(5a+3)x}{\sin 3x}\,dx - \int_0^{\frac{\pi}{5}} \frac{\sin(5a-3)x}{\sin 3x}\,dx = 0$

$$LHS = \int_0^{\frac{\pi}{5}} \left(\frac{\sin(5a+3)x}{\sin 3x} - \frac{\sin(5a-3)x}{\sin 3x}\right) dx$$

$$= \int_0^{\frac{\pi}{5}} 2\cos(5ax)\,dx \quad (\text{from (i)}) \quad ①$$

$$= \frac{2}{5a}\left[\sin(5ax)\right]_0^{\frac{\pi}{5}}$$

$$= \frac{2}{5a}\left(\sin(a\pi) - \sin 0\right)$$

$$= \frac{2}{5a}(0) \quad ①$$

$$= 0$$

$$= RHS$$

$$\therefore \int_0^{\frac{\pi}{5}} \frac{\sin(5a+3)x}{\sin 3x}\,dx = \int_0^{\frac{\pi}{5}} \frac{\sin(5a-3)}{\sin 3x}\,dx$$

Alternative:

$$LHS = \int_0^{\frac{\pi}{5}} \left(2\cos 5ax + \frac{\sin(5ax-3x)}{\sin 3x}\right) dx \quad (\text{from (i)})$$

$$= \frac{2}{5a}\left[\sin 5ax\right]_0^{\pi/5} + \int_0^{\pi/5} \frac{\sin(5a-3)x}{\sin 3x}\,dx$$

$$= \frac{2}{5a}\left(\sin a\pi - \sin 0\right) + \int_0^{\pi/5} \frac{\sin(5a-3)x}{\sin 3x}\,dx$$

$$= \int_0^{\pi/5} \frac{\sin(5a-3)x}{\sin 3x}\,dx$$

$$= RHS$$

**Questions 15**

**(b)** In a certain sequence $T_n$, $T_1 = 3, T_2 = 5$ and $T_{n+2} = 4T_{n+1} - 3T_n$. **3**

Prove by mathematical induction that $T_n = 3^{n-1} + 2$.

Step 1: Prove true for $n=1$ and $n=2$

For $n=1$, LHS $= T_1 = 3$, RHS $= 3^{1-1}+2 = 3 =$ LHS $\therefore$ True for $n=1$

For $n=2$, LHS $= T_2 = 5$, RHS $= 3^{2-1}+2 = 5 =$ LHS $\therefore$ True for $n=2$ (1)

* Must prove true for $n=1$ and $2$. $T_3 = 4T_2 - 3T_1$ will not be true if $T_1$ and $T_2$ are false.

Step 2: Assume true for $n=k$ and $n=k+1$

ie. $T_k = 3^{k-1}+2$ and

$T_{k+1} = 3^k + 2$

* Must assume true for $T_k$ and $T_{k+1}$ to prove true for $T_{k+2}$.

Prove true for $n=k+2$ (1)

ie. $T_{k+2} = 3^{k+1}+2$

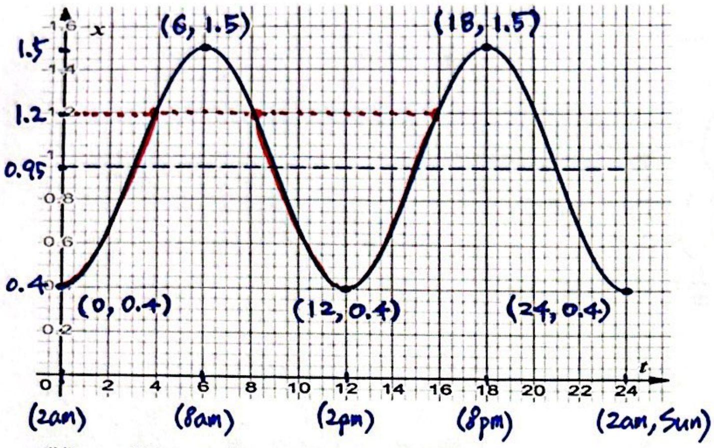
$$
\begin{aligned}
\text{LHS} &= T_{k+2} \\
&= 4T_{k+1} - 3T_k \\
&= 4(3^k+2) - 3(3^{k-1}+2) \quad (\text{by the assumption}) \\
&= 4(3^k) + 8 - 3^k - 6 \\
&= 3(3^k) + 2 \\
&= 3^{k+1} + 2 \quad (1) \\
&= \text{RHS}
\end{aligned}
$$

Step 3: By the principle of mathematical induction, the statement is true for all integer $n \geq 1$.

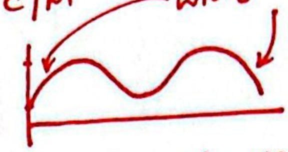
**Questions 15**

**(c)** The clearance for shipping under the Sydney Harbour Bridge is 45 metres. The Penrith Sun cruise ship is a luxury cruise ship which requires 43.8 metres height above the water level to safely cruise under the bridge. The process of sailing under the bridge will take 15 minutes. The first low tide on Saturday the 26th of August was at 2 am and the first high tide was at 8 am. The depth of water at low tide was 0.4 m and at high tide it was 1.5 m. Assume that the tidal motion is simple harmonic motion.

**(i)** Neatly draw the displacement graph showing two complete wavelengths for the Penrith Sun cruise ship starting from 2am on Saturday the 26th of August. **2**




C/M




* Must show the vertical intercept.

* Show the coordinates of the turning points.

**(ii)** Hence show that the displacement equation can be written in the form: **2**

$x = -b\cos(nt) + c$.

$T = 12$, amplitude $= \frac{1.5 - 0.4}{2} = 0.55$, centre $= 0.4 + 0.55 = 0.95$ (1)

$\frac{2\pi}{n} = 12$

$n = \frac{\pi}{6}$

$\therefore x = -0.55\cos\frac{\pi}{6}t + 0.95$ (1)

Poorly done

**(iii)** Determine between what times on Saturday can the Penrith Sun cruise ship safely make the passage under the bridge, given that no ships are allowed under the Sydney Harbour Bridge between 8 pm and 2 am? **3**

$45 - 43.8 = 1.2$

When $x = 1.2$, $t = ?$

$1.2 = -0.55\cos\frac{\pi}{6}t + 0.95$ (1)

$0.55\cos\frac{\pi}{6}t = -0.25$

$\cos\frac{\pi}{6}t = -\frac{5}{11}$

$\frac{\pi}{6}t = \pi - 1.099,\ \pi + 1.099,\ 3\pi - 1.099$

$= 2.043,\ 4.241,\ 8.326$

$t = 3.901,\ 8.099,\ 15.901$ (1)

$= 3h\ 54min,\ 8h\ 6min,\ 15h\ 54min$

$\therefore$ Between 2am and 5:54am or between 10:06 am and 5:54pm. (1)

Question 16

**(a)** What is the projection vector of the vector $2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$ on the line joining the points $(3, 4, 2)$ and $(5, 6, 3)$? **2**

$A(3,4,2)$, $B(5,6,3)$

$$\vec{AB} = \begin{pmatrix} 5-3 \\ 6-4 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \ (= \underset{\sim}{b} \text{ say})$$

$$\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$

$$|\underset{\sim}{b}|^2 = 2^2+2^2+1^2 = 9$$

$$\text{Proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \underset{\sim}{b}$$

$$= \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}}{\quad} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{4+6-6}{9} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{4}{9} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

comments: Majority of students did it well.
- common mistakes using incorrect formula, using $|\underset{\sim}{b}|$ instead of $|\underset{\sim}{b}|^2$.

**(b) (i)** If $z = \cos\theta + i\sin\theta$, show that $\sin n\theta = \frac{1}{2i}(z^n - \frac{1}{z^n})$. **2**

$$z = \cos\theta + i\sin\theta$$

$$z^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta$$

(D'Moivre theo)

$$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta$$

$$\begin{pmatrix} \cos(-\theta) = \cos\theta \to \text{even} \\ \sin(-\theta) = -\sin\theta \to \text{odd} \end{pmatrix}$$

$$\therefore z^n - z^{-n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$

$$= 2i\sin n\theta$$

$$\therefore \sin n\theta = \frac{1}{2i}(z^n - z^{-n})$$

comments: students lost 1 mark for not writing symmetry properties and D'Moivre theo.

**(ii)** Express $sin^5\theta$ in terms of multiple angles. **2**

$$(z - z^{-1})^5 = {}^5C_0 z^5 - {}^5C_1 z^4 \times z^{-1} + {}^5C_2 z^3 z^{-2} - {}^5C_3 z^2 \times z^{-3} + {}^5C_4 z \times z^{-4}$$
$$- {}^5C_5 z^{-5}$$

$$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

comments: Poor attempt, students were writing in $\cos\theta$, algebraic error, did not read the question carefully (different angles).

(iii) Hence find $\int \sin^5\theta \, d\theta$ 1

$$I = \int \left( \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta \right) d\theta$$

$$= \frac{1}{16}\left(\frac{-\cos 5\theta}{5}\right) - \frac{5}{16}\left(-\frac{\cos 3\theta}{3}\right) + \frac{5}{8}(-\cos\theta) + C$$

$$= \frac{-\cos 5\theta}{80} + \frac{5\cos 3\theta}{48} - \frac{5}{8}\cos\theta + C$$

Comments : students were messing up positive / negative signs, making integration very complex.

(c) Evaluate exactly $\int_{\sqrt{3}}^{3} \frac{6x+18}{x^2+9} \, dx$ 3

$$\int_{\sqrt{3}}^{3} \frac{6x+18}{x^2+9} dx = \int_{\sqrt{3}}^{3} \frac{2(3x)}{x^2+9} dx + \int_{\sqrt{3}}^{3} \frac{18}{x^2+9} dx$$

$$= 2\left[\ln|x^2+9|\right]_{\sqrt{3}}^{3} + \frac{18}{3}\left[\tan^{-1}\frac{x}{3}\right]_{\sqrt{3}}^{3}$$

$$= 2\left[\ln|18| - \ln|12|\right] + 6\left[\tan^{-1}\frac{3}{3} - \tan^{-1}\frac{\sqrt{3}}{3}\right]$$

$$= 2\ln\left|\frac{18}{12}\right| + 6\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$$

$$= 2\ln\frac{3}{2} + 6 \times \frac{\pi}{12}$$

$$= 2\ln\frac{3}{2} + \pi/2$$

Comments : students did well in this question, Few students made algebraic errors in the second set of integrand, forgot 6.

(d) (i) Prove that $\cos x = 1 - 2sin^2\frac{x}{2}$ 1

I : $\cos 2x = 1 - 2\sin^2 x$

writing $x \to \frac{x}{2}$

$\cos 2\left(\frac{x}{2}\right) = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$\therefore \cos x = 1 - 2\sin^2 x/_2$

II : $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$= \cos\frac{x}{2}\cos\frac{x}{2} - \sin\frac{x}{2}\sin\frac{x}{2}$

$= \cos^2 x/_2 - \sin^2 x/_2$ (double angle formula)

$= (1 - \sin^2 x/_2) - \sin^2 x/_2$

$= 1 - 2\sin^2 x/_2$

comments : Majority of students got this right. However, it was disappointed to see some students were making this 1 mark easy question into a long / complex question.

(ii) Prove by mathematical induction that: 5

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \text{ for all } n \in Z^+.$$

Step 1 for $n=1$, $LHS = \frac{1}{2} + \cos x$

$RHS = \dfrac{\sin(1+\frac{1}{2})x}{2\sin\frac{x}{2}}$

$= \dfrac{\sin x\cos\frac{x}{2} + \cos x\sin\frac{x}{2}}{2\sin x/_2}$

$= \dfrac{(2\sin x/_2\cos x/_2)\cos x/_2 + \cos x\sin x/_2}{2\sin x/_2}$

$= \cos^2 x/_2 + \frac{1}{2}\cos x$

$= \dfrac{2(1 - \sin^2 x/_2) + \cos x}{2}$

$= \dfrac{1 + \cos x + \cos x}{2}$

$= \frac{1}{2} + \cos x$

$LHS = RHS$

the result is true for $n=1$

comments :

- students did not do a good job for showing for $n=1$
- A couple of students actually showed for $n=0$.
- students are strongly suggested to revise double-angle, half-angle formulae.

Step 2 : Let the result be true for $n=k$, $k \in Z^+$

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kx = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

Step 3 To show that the result is true for $n=k+1$,

i.e. $$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kx + \cos(k+1)x = \frac{\sin\left((k+1)+\frac{1}{2}\right)x}{2\sin x/2}$$

$$= \frac{\sin\left(\left(k+\frac{3}{2}\right)x\right)}{2\sin x/2}$$

LHS: $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kx + \cos(k+1)x$

$$= \left(\frac{1}{2} + \cos x + \cdots + \cos kx\right) + \cos(k+1)x$$

(using assumption)

$$= \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin x/2} + \cos(k+1)x$$

$$= \frac{\sin\left(k+\frac{1}{2}\right)x + 2\sin x/2 \cos(k+1)x}{2\sin x/2}$$

$$= \frac{\sin\left(k+\frac{1}{2}\right)x + \sin\left(\frac{x}{2}+(k+1)x\right) + \sin\left(\frac{x}{2} - kx - x\right)}{2\sin x/2}$$

(products to sums)

$$= \frac{\sin\left(k+\frac{1}{2}\right)x + \sin\left(\left(k+\frac{3}{2}\right)x\right) + \sin\left(-kx-\frac{x}{2}\right)}{2\sin x/2}$$

$\sin(-\theta) = -\sin\theta$

$$= \frac{\cancel{\sin\left(k+\frac{1}{2}\right)x} + \sin\left(\left(k+\frac{3}{2}\right)x\right) - \cancel{\sin\left(k+\frac{1}{2}\right)x}}{2\sin x/2}$$

$$= \frac{\sin\left(k+3/2\right)x}{2\sin x/2}$$

$= RHS$

Comments:
- Very poor attempt
- students are suggested to revisit the topic

Hence, using the Principle of Mathematical Induction, the result is true $\forall k \in Z^+$